Expanding Annular Domain Algorithm to Estimate Domains of Attraction for Power System Stability Analysis

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***Abstract*—This paper presents an Expanding Annular Domain (EAD) algorithm combined with Sum of Squares (SOS) programming to estimate and maximize the domain of attraction (DA) of power systems. The proposed algorithm can systematically construct polynomial Lyapunov functions for power systems with transfer conductance and reliably determine a less conservative approximated DA, which are quite difficult to achieve with traditional methods. With linear SOS programming, we begin from an initial estimated DA, then enlarge it by iteratively determining a series of so-called annular domains of attraction, each of which is characterized by level sets of two successively obtained Lyapunov functions. Moreover, the EAD algorithm is theoretically analyzed in detail and its validity and convergence are shown under certain conditions. In the end, our method is tested on two classical power system cases and is demonstrated to be superior to existing methods in terms of computational speed and conservativeness of results.**

***Index Terms*—Domain of attraction, Lyapunov functions, power system transient stability, sum of squares programming.**

I. INTRODUCTION

WE consider the problem of estimating the domain of attraction (DA) of power systems to assess transient stability [1]. Domain of attraction, a key quantitative measure of transient stability, is defined as an invariant set such that all trajectories starting from points in this set converge to a corresponding asymptotically stable equilibrium point (ASEP) [2]. If the initial state of a post-fault system is located in the DA, the power system will be deemed transient stable. It no longer needs to track subsequent state trajectory, which will facilitate rapid evaluations of post-fault system security and designs of effective stabilization controllers [3], [4].

Due to difficulty of calculating the exact DA, research has focused on determining Lyapunov functions whose level sets characterize estimates of the DA. However, when applied to power systems, traditional methods [5]–[8] based on Lyapunov functions have encountered two difficulties, i.e, the difficulty of systematically constructing Lyapunov functions considering transfer conductance and the difficulty of reliably determining a sufficiently large approximated DA. For instance, the energy functions method, which has been extensively investigated, fails to obtain an analytical energy function for power systems with transfer conductance because of the path-dependent integral terms involved [5]. Although a numerical energy function can be computed by using the ray (or trapezoidal) approximation schemes to numerically approximate the path-dependent terms, it is not a well-defined function and has inevitable calculation errors [6], [9]. Moreover, the energy functions method requires reliable computation of the critical energy value to guarantee accuracy of the estimated boundary of the DA, which is a challenging task. There are many approaches to calculate the critical energy value such as the Closest Unstable Equilibrium Point (UEP) approach [10], controlling UEP approach [11], PEBS approach [12] and BCU approach [13]. But most of them can only estimate the local relevant boundary of the DA except the Closest UEP approach. In addition to the energy function method, another method based on Popov stability criterion can systematically construct a Lur'e-Postnikov type Lyapunov function to obtain an estimated DA, but it requires satisfaction of sector conditions and will fail if the transfer conductance is not negligible [7]. Besides, [8] proposes a method using the extended LaSalle's invariance principle to construct an extended Lyapunov function, whose derivative is allowed to be positive in some bounded regions included in the DA, for power systems with small transfer conductance. However, it is rather difficult for this method to determine the special regions involved and to analyze practical cases with relatively large transfer conductance.

The Sum of Squares (SOS) technique, first introduced by Parrilo in 2000 [14], provides a novel approach to estimate the DA of power systems by solving semidefinite programs with matrix inequality constraints [15]–[17]. This technology is able to overcome the aforementioned two difficulties faced by traditional methods. As shown in [18], the authors transformed the non-negative conditions related to the DA, which are based on the Lyapunov's direct method, into an SOS optimization problem to systematically construct local Lyapunov functions and maximize the estimate of the DA. To solve the SOS optimization problem with bilinear constraints, an algorithm containing two iterative loops and four linear SOS optimization problems, called expanding interior algorithm, has been

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proposed [18]. Since this algorithm is too complicated for practice application, it is improved by reducing the number of the linear SOS problems solved at each iteration to half [19]. The algorithm has been successfully applied to the reduced-order model of a two-machine-infinite-bus system [18], [19]. It was further applied to the detailed model of a single-machine-infinite-bus system, considering voltage dynamics and both voltage and frequency regulators [20]. Overall, such an SOS-based method is promising, but the existing improved algorithm is still too complex for power systems and may lead to conservative results.

The main contribution of this paper is twofold. First, we propose an Expanding Annular Domain (EAD) algorithm combined with Sum of Squares programming to enlarge a provable DA of power systems. The algorithm is developed from the modified Lyapunov stability theory introduced in Section II and is totally different from the existing expanding interior algorithm based on the conventional Lyapunov stability theory. Compared with the latter algorithm, the EAD algorithm achieves faster calculation and less conservative results by simplifying iteration steps and relaxing the stability criteria. Meanwhile, it is capable of addressing the two difficulties encountered by traditional methods. Second, we theoretically analyze the details of the proposed algorithm, including proving the necessity of some constraints, showing the feasibility of some iterative steps and confirming the convergence of the algorithm.

To the best of our knowledge, the concepts of the EAD algorithm have never been applied to transient stability analysis of power systems. Finally, the effectiveness of our method is demonstrated on a two-machine-infinite-bus system and the IEEE 4-machine-11-bus system.

Notation: Let $\mathbb{R}$, $\mathbb{R}_+$ and $\mathbb{Z}_+$ be the set of real numbers, positive real numbers and positive integers, respectively. The set of $n\times m$ real matrices is represented by $\mathbb{R}^{n\times m}$. The set of polynomials and SOS polynomials in $x\in\mathbb{R}^n$ are denoted by $\mathbb{P}_n[x]$ and $\mathbb{S}_n[x]$, respectively. Moreover, we use $\mathbb{P}_n^{M\times N}[x]$ and $\mathbb{S}_n^{M\times N}[x]$ to represent the set of $M\times N$ polynomials matrices and SOS polynomials matrices in $x\in\mathbb{R}^n$, respectively. Additionally, the boundary of a set D is represented by ∂D. Finally, $\deg(p_1,p_2,\ldots,p_k)$ denotes the maximum degree of all polynomials in the set $\{p_1,p_2,\ldots,p_k\}$.

II. Preliminaries and Formulations

A. Modified Lyapunov Stability Theory

Consider an autonomous nonlinear dynamic system governed by

$$\dot{x}(t)=f(x(t)) \tag{1}$$

where $x\in\mathbb{R}^n$ is the state vector, $f:\mathbb{R}^n\to\mathbb{R}^n$ is the polynomial vector field. We suppose the origin $x=\mathbf{0}$ is an equilibrium point, i.e., $f(\mathbf{0})=\mathbf{0}$. If the origin is an asymptotically stable equilibrium point (ASEP), the corresponding domain of attraction (DA) is defined as $\mathcal{D}=\{x(0)\in\mathbb{R}^n|\lim_{t\to\infty}x(t;x(0))=0\}$.

According to the conventional Lyapunov stability theory [2], if there exist an open set $\Omega\subseteq\mathbb{R}^n$ and a continuously differentiable function $V(x):\Omega\to\mathbb{R}$, called Lyapunov function, such that $V(\mathbf{0})=0, V(x)>0,\ \forall x\in\Omega\backslash\{\mathbf{0}\}$ and $\dot{V}(\mathbf{0})=0,\dot{V}(x)<0,\ \forall x\in\Omega\backslash\{\mathbf{0}\}$, then a set $D=\{x|V(x)\le c, c>0\}\subset\Omega$ is guaranteed to be an invariant subset of the DA with respect to the origin. To relax the aforementioned conditions, we will introduce the modified Lyapunov stability theory.

Definition 1: For a given system (1) with an open set D_1 and a bounded set D_2 satisfying that $\mathbf{0}\in D_1\subset D_2\subseteq\mathbb{R}^n$, set $A=D_2\backslash D_1$ is regarded as an annular domain of attraction if every solution $x(t;x(0))$ of system (1) starting in A will remain in D_2 and enter into D_1 as $t\to t_1, t_1\in\mathbb{R}_+$.

Theorem 1: For a given system (1), let $V(x):\mathbb{R}^n\to\mathbb{R}$ be a continuously differentiable function. Consider an open set D_1 and a bounded set $D_2=\{x\in\mathbb{R}^n|V(x)\le 1\}$ satisfying $\mathbf{0}\in D_1\subset D_2\subseteq\mathbb{R}^n$. Then set $A=D_2\backslash D_1$ is an annular domain of attraction if the following condition holds:

$$\begin{aligned}&V(\mathbf{0})=0,\ V(x)>0,\ \forall x\neq\mathbf{0}\\ &\dot{V}(x)=\frac{\mathrm{d}V}{\mathrm{d}x}f(x)<0,\ \forall x\in A.\end{aligned} \tag{2}$$

Proof: It is clear that $D_2=\{x\in\mathbb{R}^n|V(x)\le 1\}$ is a compact set. Then, any trajectory starting in D_2 at $t=0$ stays in D_2 for all $t\ge 0$ since

$$\dot{V}<0,\ \forall x\in A\subset D_2\Rightarrow V(x(t))\le 1,\ \forall t>0,\ \forall x(0)\in A$$

To show that every trajectory starting in A will enter into D_1 in finite time, we use a contradiction argument. Suppose that $\exists x(0)\in A$ such that for all $t\ge 0$, $x(t;x(0))\notin D_1$. Let $-\gamma=\max_{x\in A}\dot{V}(x)$, which exists because the continuous function $\dot{V}(x)$ has a maximum over the compact set A. By (2), $-\gamma<0$. It follows that $V(x(t))=V(x(0))+\int_0^{\mathrm{t}}\dot{V}(x(\tau))d\tau\le V(x(0))-\gamma t$. Since the right-hand side will eventually become negative, the inequality contradicts the fact that for all $x\neq\mathbf{0}$, $V(x)>0$. Thus, there exists a $t_1>0$ such that $x(t_1;x(0))\in D_1$. From what has been discussed above, based on Definition 1, we can regard A as an annular domain of attraction. ■

Remark 1: Theorem 1 implies that if $A=D_2\backslash D_1$ is an annular domain of attraction for a given system (1), then D_2 is a positively invariant set because $x(t)\in D_2$ for all $x(0)\in D_2$ and $t\ge 0$.

Theorem 2: Let $x=\mathbf{0}$ be an ASEP for system (1) and let a set D_1 containing the origin be an open invariant subset of the DA. Consider a bounded set $D_2=\{x\in\mathbb{R}^n|V(x)\le 1\}$ satisfying $D_1\subset D_2$, where $V(x):\mathbb{R}^n\to\mathbb{R}$ is a continuously differentiable function. If the set $A=D_2\backslash D_1$ is an annular domain of attraction, then the set D_2 is an invariant subset of the DA with respect to the origin.

Proof: If $A=D_2\backslash D_1$ is an annular domain of attraction, then every trajectory starting in A will remain in D_2 and enter into D_1 eventually. Since D_1 is the subset of the DA with respect to the origin, the trajectory will approach to the origin as $t\to\infty$. From remark 1, we know that D_2 is positively invariant. Thus, D_2 is an invariant subset of the DA. ■

Remark 2: An invariant subset of the DA can be regarded as an estimate of the DA. Theorem 2 is a relaxation of the conventional Lyapunov stability theory since the derivative of the Lyapunov function $V(x)$ in Theorem 2 only needs to be negative in $A = D_2 \backslash D_1$, rather than in D_2. Therefore, when expanding estimated DAs, Theorem 2 requires less stringent stability criteria and thus produces less conservative results compared with conventional Lyapunov stability theory.

B. Sum of Squares Programming

Definition 2 [16]: A multivariate polynomial $p(x)$ in $x \in \mathbb{R}^n$ is a sum of squares (SOS), if there exist polynomials $f_1(x), \ldots, f_m(x)$ such that $p(x) = \sum_{i=1}^{m} f_i^2(x)$.

Clearly, $p(x)$ being an SOS naturally implies $p(x) \geq 0$ for all $x \in \mathbb{R}^n$. However, the converse is not always true [21]. The SOS programming problem is a convex optimization problem with one or more SOS constraints (that is, some polynomials are required to be SOS polynomials). When establishing an SOS programming problem, we often use the *Positivstellensatz theorem* to determine positive semi-definiteness of a polynomial over semi-algebraic sets, as shown below.

Theorem 3 (Positivstellensatz Theorem) [17], [22]: Consider three sets of polynomials $\mathcal{M}(l_1, \ldots, l_t) = \{l_1^{k_1} l_2^{k_2} \cdots l_t^{k_t} | l_1, \ldots, l_t \in \mathbb{P}_n[x], k_1, \ldots, k_t \in \{0, 1, 2, \ldots\}\}$, $\mathcal{C}(p_1, \ldots, p_s) = \{s_0 + \sum s_i b_i | s_i \in \mathbb{S}_n[x], b_i \in \mathcal{M}(p_1, \ldots, p_s), p_1, \ldots, p_s \in \mathbb{P}_n[x]\}$ and $\mathcal{I}(g_1, \ldots, g_u) = \{\sum \lambda_k g_k | \lambda_k \in \mathbb{P}_n[x], g_1, \ldots, g_u \in \mathbb{P}_n[x]\}$. Then the set

$$\left\{ x \in \mathbb{R}^n \;\middle|\; \begin{array}{l} p_1(x) \geq 0, \ldots, p_s(x) \geq 0 \\ g_1(x) = 0, \ldots, g_u(x) = 0 \\ l_1(x) \neq 0, \ldots, l_t(x) \neq 0 \end{array} \right\} \tag{3}$$

is empty, if and only if there exist polynomials $p \in \mathcal{C}(p_1, \ldots, p_s)$, $g \in \mathcal{I}(g_1, \ldots, g_u)$ and $l \in \mathcal{M}(l_1, \ldots, l_t)$ such that

$$p + g + l^2 = 0 \tag{4}$$

To illustrate the application of Theorem 3, we consider a brief example. Given a semi-algebraic set condition $S = \{x \in \mathbb{R}^n \mid p_1(x) \geq 0, p_2(x) \geq 0, g_1(x) = 0, l_1(x) \neq 0\} = \varnothing$, we can choose relatively simple polynomials $p = s_0 + s_1 p_1 + s_2 p_2$, $g = \lambda_1 g_1$ and $l = l_1$ such that $p + g + l^2 = s_0 + s_1 p_1 + s_2 p_2 + \lambda_1 g_1 + l_1^2 = 0$, where $s_{0,1,2} \in \mathbb{S}_n[x]$ and $\lambda_1 \in \mathbb{P}_n[x]$, according to Theorem 3. Then $-(s_1 p_1 + s_2 p_2 + \lambda_1 g_1 + l_1^2) = s_0 \in \mathbb{S}_n$ holds, which completes the transformation from a semi-algebraic set condition $S = \varnothing$ to an SOS constraint.

Furthermore, an SOS programming problem can be transformed into a semidefinite program (SDP) [14] and can be efficiently solved by SDP techniques. However, SDP techniques have worse-case polynomial time complexity. As the degree of a polynomial or its number of variables is increased, the corresponding SOS programming will have increasing computational complexity.

All SOS programs formulated in this paper were efficiently solved by the MATLAB toolboxes YALMIP [23] and a semidefinite programming solver MOSEK [24].

C. Power Systems Description and Coordinate Transformation

We consider the internal node model [4] of an n-machine power system expressed as:

$$\begin{bmatrix} \dot{\delta}_{1n} \\ \dot{\delta}_{2n} \\ \vdots \\ \dot{\delta}_{n-1,n} \\ \dot{\omega}_1 \\ \vdots \\ \dot{\omega}_n \end{bmatrix} = \begin{bmatrix} \omega_1 - \omega_n \\ \omega_2 - \omega_n \\ \vdots \\ \omega_{n-1} - \omega_n \\ 1/M_1(P_{\mathrm{m}1} - P_{\mathrm{e}1} - D_1\omega_1) \\ \vdots \\ 1/M_n(P_{\mathrm{m}n} - P_{\mathrm{e}n} - D_n\omega_n) \end{bmatrix} \tag{5}$$

where the electrical power is denoted by $P_{\mathrm{e}i} = \sum_{j=1}^{n} E_i E_j [B_{ij} \sin(\delta_{in} - \delta_{jn}) + G_{ij} \cos(\delta_{in} - \delta_{jn})]$, $i = 1, 2, \ldots, n-1$; δ_{in} represents the relative rotor angle with respect to the nth generator and ω_i is the rotor angular velocity; M_i, D_i, $P_{\mathrm{m}i}$ are the inertia coefficient, damping coefficient and mechanical input, respectively; E_i represents the internal voltage; B_{ij}, G_{ij} are the susceptance and conductance between generators i and j, respectively. For simplicity, we need to shift the equilibrium point of the system (5) to the origin. Hence, assuming that $(\delta_{1n}^*, \delta_{2n}^*, \ldots, \delta_{n-1,n}^*, 0, \ldots, 0)$ is the stable equilibrium point, let the new state variable vector $x \in \mathbb{R}^{2n-1}$ be $x = [\Delta\delta_{1n}, \ldots, \Delta\delta_{n-1,n}, \omega_1, \ldots, \omega_n] = [\delta_{1n} - \delta_{1n}^*, \ldots, \delta_{n-1,n} - \delta_{n-1,n}^*, \omega_1, \ldots, \omega_n]$.

However, SOS programming cannot be directly applied to the power system (5) because it is not expressed as a polynomial system. Thus, [18] has proposed a coordinate transformation $z = h(x)$, where $h : \mathbb{R}^{2n-1} \to \mathbb{R}^{3n-2}$ is the function such that

$$\begin{cases} z_{2i-1} = \sin x_i \\ z_{2i} = 1 - \cos x_i, \; i = 1, 2, \ldots, n-1 \\ z_{2(n-1)+j} = x_{n-1+j}, \; j = 1, 2, \ldots, n \end{cases} \tag{6}$$

From (5) and (6), the transformed system is described by

$$\begin{cases} \dot{z} = F(z) \\ G(z) = 0 \end{cases} \tag{7}$$

where $z \in \mathbb{R}^M (M = 3n-2)$ is the state vector, $F : \mathbb{R}^M \to \mathbb{R}^M$ and $G : \mathbb{R}^M \to \mathbb{R}^N (N = n-1)$ are polynomial functions. System (7) can be rewritten as:

$$\begin{cases} \begin{bmatrix} \dot{z}_1 \\ \dot{z}_2 \\ \dot{z}_3 \\ \dot{z}_4 \\ \vdots \\ \dot{z}_{2n-3} \\ \dot{z}_{2n-2} \\ \dot{z}_{2n-1} \\ \vdots \\ \dot{z}_{3n-2} \end{bmatrix} = \begin{bmatrix} (1 - z_2)(z_{2n-1} - z_{3n-2}) \\ z_1(z_{2n-1} - z_{3n-2}) \\ (1 - z_4)(z_{2n} - z_{3n-2}) \\ z_3(z_{2n} - z_{3n-2}) \\ \vdots \\ (1 - z_{2n-2})(z_{3n-3} - z_{3n-2}) \\ z_{2n-3}(z_{3n-3} - z_{3n-2}) \\ 1/M_1(P_{\mathrm{m}1} - P_{\mathrm{e}1,z} - D_1 z_{2n-1}) \\ \vdots \\ 1/M_n(P_{\mathrm{m}n} - P_{\mathrm{e}n,z} - D_n z_{3n-2}) \end{bmatrix} \\ \begin{bmatrix} z_1^2 + z_2^2 - 2z_2 \\ z_3^2 + z_4^2 - 2z_4 \\ \vdots \\ z_{2n-3}^2 + z_{2n-2}^2 - 2z_{2n-2} \end{bmatrix} = 0 \end{cases} \tag{8}$$

where

$$\begin{aligned}P_{\mathrm{ei},z} = &\sum_{j=1}^{n-1} E_iE_j[B_{ij}(\alpha_{ij}p_{ij}+\beta_{ij}q_{ij})\\&+G_{ij}(\alpha_{ij}q_{ij}-\beta_{ij}p_{ij})]\\&+E_iE_n[B_{in}(z_{2i-1}\cos\delta_{in}^*+(1-z_{2i})\sin\delta_{in}^*)\\&+G_{in}((1-z_{2i})\cos\delta_{in}^*-z_{2i-1}\sin\delta_{in}^*)]\\ p_{ij} =& z_{2i-1}(1-z_{2j})-(1-z_{2i})z_{2j-1}\\ q_{ij} =& (1-z_{2i})(1-z_{2j})+z_{2i-1}z_{2j-1}\\ \alpha_{ij} =& \cos(\delta_{in}^*-\delta_{jn}^*), \beta_{ij}=\sin(\delta_{in}^*-\delta_{jn}^*)\\ i =& 1,2,\ldots,n-1. \end{aligned} \tag{9}$$

Note that, by the coordinate transformation, the power system (5) can be described by polynomial overdetermined Differential Algebraic equations (7), whose equality constraints restrict the states $z \in \mathbb{R}^{3n-2}$ to the original state manifold $x \in \mathbb{R}^{2n-1}$.

III. Estimating the DA of Power Systems

In this section, an EAD algorithm combined with SOS programming will be proposed to estimate the DA of power systems (7). First, we compute an initial polynomial Lyapunov function and obtain a rough estimate of the DA. Then, the expanding annular domain algorithm is introduced to enlarge the initial estimated DA.

A. Initialize the Estimation of the DA

For power systems, the DA of the ASEP at origin can be estimated by a level set of a Lyapunov function $V(z)$, i.e.

$$\mathcal{L}_c(V(z)) = \{z \in \mathbb{R}^M | V(z) \le c, G(z)=0, c>0\}. \tag{10}$$

Meanwhile, we define an open level set of $V(z)$ as:

$$\mathbb{L}_c(V(z)) = \{z \in \mathbb{R}^M | V(z) < c, G(z)=0, c>0\}. \tag{11}$$

Based on SOS programming and Theorem 3, an initial polynomial Lyapunov function $V_0(z)$ can be obtained as following. Given a semi-algebraic domain $P=\{z\in\mathbb{R}^M | p(z)\le\gamma, \gamma>0\}$ containing the origin, where $p(z)$ is a positive definite polynomial, we search for a function $V_0(z)$ with $V_0(0)=0$ such that

$$V_0(z) > 0 \quad \forall z \in \{z\in\mathbb{R}^M | G(z)=0\}\backslash\{0\} \tag{12a}$$

$$\dot{V}_0(z) < 0 \quad \forall z \in \{z\in\mathbb{R}^M | \gamma - p(z)\ge 0, G(z)=0\}\backslash\{0\} \tag{12b}$$

then $V_0(z)$ is a polynomial Lyapunov function of (7). Further, we replace $z\neq 0$ with $q_{1,2}(z)\neq 0$, where $q_{1,2}$ are given SOS polynomials with small coefficients, such as $q_{1,2} = 1\times 10^{-6}\sum_{i=1}^M z_i^d$. Different choices of $q_{1,2}$ have little effect on the result because $q_{1,2}$ are sufficiently small. Then we formulate the conditions (12) as the set emptiness conditions:

$$\{z\in\mathbb{R}^M \mid G(z)=0, q_1(z)\neq 0,\ V_0(z)\le 0\}=\varnothing \tag{13a}$$

$$\{z\in\mathbb{R}^M \mid \gamma-p(z)\ge 0,\ G(z)=0,\ q_2(z)\neq 0, \dot{V}_0(z)\ge 0\} = \varnothing \tag{13b}$$

According to Theorem 3, we obtain the following SOS programming problem:

$$\begin{aligned}&(\text{SOSP0}) \quad \underset{\substack{V_0\in\mathbb{P}_M[z],V_0(0)=0,\\ s_1\in\mathbb{S}_M[z],\lambda_1,\lambda_2\in\mathbb{P}_M^N[z]}}{\text{search}} V_0\\ &\text{s.t.}\\ &V_0-\lambda_1^{\mathrm{T}}G-q_1\in\mathbb{S}_M[z] \quad (14\text{a})\\ &-s_1(\gamma-p)-\dot{V}_0-\lambda_2^{\mathrm{T}}G-q_2\in\mathbb{S}_M[z] \quad (14\text{b})\end{aligned}$$

Note that different choices of $p(z)$ and γ will lead to different results of V_0. Generally, it is necessary to ensure that domain $P=\{z|p(z)\le\gamma\}$ is small enough to be contained within the DA. If the subsequent EAD algorithm can not get satisfactory results, $p(z)$ and γ can be re-selected to initialize the V_0.

After finding a feasible solution $V_0(z)$, we try to get a maximum level set $\mathcal{L}_c(V_0(z))$ such that $\dot{V}_0(z)<0$ for all $z\in\mathcal{L}_c(V_0(z))\backslash\{0\}$. $\mathcal{L}_c(V_0(z))$ can be regarded as a rough estimate of the DA. Similarly, this can be formulated as an SOS programming problem:

$$\begin{aligned}&(\text{SOSP0'}) \quad \max_{s_1,s_2\in\mathbb{S}_M[z],\lambda\in\mathbb{P}_M^N[z]} c\\ &\text{s.t.}\\ &-s_1(c-V_0)-s_2\dot{V}_0-\lambda^{\mathrm{T}}G-q_1\in\mathbb{S}_M[z] \quad (15)\end{aligned}$$

where V_0 is obtained from SOSP0. We can efficiently solve SOSP0' using a bisection search on c. Given an arithmetic sequence $\{c^{(1)},c^{(2)},\ldots,c^{(i)},c^{(j)},\ldots\}$, where $0<c^{(1)}<c^{(2)}<\cdots$, we assume $c^{(i)}$ makes the constraint in SOSP0' feasible and $c^{(j)}$ does the opposite, which are denoted by $S(c^{(i)})<0$ and $S(c^{(j)})>0$, respectively. Then in the interval $[c^{(i)},c^{(j)}]$, the zero of the function $S(\cdot)$ can be approached by the Bisection method and can be regarded as the optimal solution of SOSP0', denoted by c_0. Then $V^{(1)}=V_0(z)/c_0$ and $D^{(1)}=\mathcal{L}_1(V^{(1)}(z))$ can be regarded as an initial Lyapunov function and the rough estimate of the DA, respectively.

B. Expanding the Estimated DA by the EAD Algorithm

Figure 1 shows the idea of the EAD algorithm, which is introduced as follows. With linear SOS programming, an initial estimated DA is expanded by iteratively determining a series of annular domains of attraction $A^{(k)}=D^{(k+1)}\backslash\mathbb{L}_\beta(V^{(k)}(z))$, where $k\in\{1,2,\ldots\}$, $\beta\in(0,1]$, $D^{(k)}=\mathcal{L}_1(V^{(k)}(z))\supseteq\mathbb{L}_\beta(V^{(k)}(z))$ is a previously estimated DA and $D^{(k+1)}=\mathcal{L}_1(V^{(k+1)}(z))$ is a candidate for the larger estimated DA. The following proposition provides theoretical guarantees for it.

Proposition 1: For a given system (7), suppose that $D^{(k)}=\mathcal{L}_1(V^{(k)}(z))$ is a known estimate of the DA. If there exist a continuously differentiable polynomial function $V^{k+1}(z)$, a bounded set $D^{(k+1)}=\mathcal{L}_1(V^{(k+1)}(z))$ and a positive number $\beta\in(0,1]$ such that

$$D^{(k)}\subset D^{(k+1)} \tag{16a}$$

$$V^{(k+1)}(0)=0,\ V^{(k+1)}(z)>0 \quad \forall z\in\{z\in\mathbb{R}^M|G(z)=0\}\backslash\{0\} \tag{16b}$$

$$\dot{V}^{(k+1)}(z)<0 \quad \forall z\in\big\{z\in\mathbb{R}^M|V^{(k+1)}(z)\le 1,$$

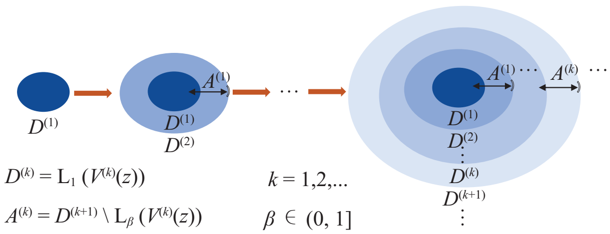

Fig. 1. Idea of the EAD algorithm.

$$V^{(k)}(z) \geq \beta, G(z) = 0\} \tag{16c}$$

then $D^{(k+1)}$ is also an estimate of the DA for system (7).

Proof: According to Theorem 1, constraints (16) imply that domain $A = D^{(k+1)} \backslash \mathbb{L}_\beta(V^{(k)}(z))$ is an annular domain of attraction. Since $D^{(k)}$ is a known estimated DA and $\mathbb{L}_\beta(V^{(k)}(z)) \subseteq D^{(k)} \subset D^{(k+1)}$, we can conclude that $D^{(k+1)}$ is also an estimate of the DA from Theorem 2. ■

Based on Proposition 1, $D^{(k)}$ can be enlarged by iteratively calculating the Lyapunov function $V^{(k+1)}(z)$. With applying Theorem 3 and SOS programming, constraints (16) can be transformed as the following SOS conditions:

$$\text{(SOSP1)} \quad \underset{\substack{V^{(k+1)} \in \mathbb{P}_M[z], V^{(k+1)}(0)=0,\\ s_1,s_2,s_3,s_4 \in \mathbb{S}_M[z], \lambda_1,\lambda_2,\lambda_3 \in \mathbb{P}_M^N[z]}}{\text{search}} \quad V^{(k+1)}$$

s.t.

$$-\, s_1(1 + \varepsilon_1 - V^{(k)}) + (1 - V^{(k+1)}) - \lambda_1^{\mathrm{T}} G \in \mathbb{S}_M[z] \tag{17a}$$

$$V^{(k+1)} - \lambda_2^{\mathrm{T}} G - q \in \mathbb{S}_M[z] \tag{17b}$$

$$\begin{aligned} &-\, s_2(1 - V^{(k+1)}) - s_3(V^{(k)} - \beta) - s_4 \dot{V}^{(k+1)} \\ &-\, \lambda_3^{\mathrm{T}} G - \varepsilon_2 \in \mathbb{S}_M[z] \end{aligned} \tag{17c}$$

where $0 < \beta \leq 1$, $V^{(k)}$ is given, $\varepsilon_{1,2} > 0$ are sufficiently small parameters, q is a given positive definite polynomial radially unbounded and can be expressed as $q = 1 \times 10^{-6} \sum_{i=1}^{M} z_i^d$ with even degree d, for example. Obviously, if the SOSP1 has a feasible solution $V^{(k+1)}$, then $D^{(k)} \subset D^{(k+1)}$ and $D^{(k+1)}$ is an estimate of the DA. However, since the problem SOSP1 has bilinear terms of the variables such as $s_2 V^{(k+1)}$ and $s_4 \dot{V}^{(k+1)}$, it cannot be efficiently solved. To address this problem, a coordinate-wise iterative method mentioned in [25] is applied here to transform the above constraints into a linear semidefinite program at each iteration, which helps to get a feasible solution for the problem SOSP1. First, we propose the following assumption.

Assumption 1: Assume that $\{z \in \mathbb{R}^M | V^{(k)}(z) \leq \beta\} \subset \{z \in \mathbb{R}^M | V^{(k+1)}(z) \leq \beta\}$, where $\beta > 0$.

If Assumption 1 is satisfied, we have $\{z | V^{(k)}(z) \geq \beta\} \supset \{z | V^{(k+1)}(z) \geq \beta\}$. Then, the constraint (16c) implies that $\dot{V}^{(k+1)}(z) < 0$ holds $\forall z \in \{z \in \mathbb{R}^M \mid V^{(k+1)}(z) \leq 1, V^{(k+1)}(z) \geq \beta, G(z) = 0, 0 < \beta \leq 1\}$. Substituting $V^{(k)}$ for $V^{(k+1)}$ yields the following new SOS condition:

$$\text{(SOSP2)} \quad \underset{s_2,s_3,s_4 \in \mathbb{S}_M[z], \lambda_3 \in \mathbb{P}_M^N[z]}{\text{search}} \quad s_2, s_4$$

s.t.

$$\begin{aligned} &-\, s_2(1 - V^{(k)}) - s_3(V^{(k)} - \beta) - s_4 \dot{V}^{(k)} \\ &-\, \lambda_3^{\mathrm{T}} G - \varepsilon_2 \in \mathbb{S}_M[z] \end{aligned} \tag{18}$$

where $V^{(k)}$ is known. Obviously, (18) is a linear SOS constraint and can be efficiently solved by using linear semidefinite programming. We save the feasible solutions of s_2, s_4 as $\bar{s}_2, \bar{s}_4$, respectively.

Secondly, replacing s_2, s_4 by $\bar{s}_2, \bar{s}_4$ in constraint (17c), we obtain a new SOS condition. Combining it with SOS conditions (17a), (17b) and considering Assumption 1, we formulate the following SOS problem:

$$\text{(SOSP3)} \quad \underset{\substack{V^{(k+1)} \in \mathbb{P}_M[z], V^{(k+1)}(0)=0,\\ s_1,s_3,s_5 \in \mathbb{S}_M[z], \lambda_1,\lambda_2,\lambda_3,\lambda_4 \in \mathbb{P}_M^N[z]}}{\text{search}} \quad V^{(k+1)}$$

s.t.

$$-\, s_1(1 + \varepsilon_1 - V^{(k)}) + (1 - V^{(k+1)}) - \lambda_1^{\mathrm{T}} G \in \mathbb{S}_M[z] \tag{19a}$$

$$V^{(k+1)} - \lambda_2^{\mathrm{T}} G - q \in \mathbb{S}_M[z] \tag{19b}$$

$$\begin{aligned} &-\, \bar{s}_2(1 - V^{(k+1)}) - s_3(V^{(k)} - \beta) - \bar{s}_4 \dot{V}^{(k+1)} \\ &-\, \lambda_3^{\mathrm{T}} G - \varepsilon_2 \in \mathbb{S}_M[z] \end{aligned} \tag{19c}$$

$$-\, s_5(\beta - V^{(k)}) + (\beta - V^{(k+1)}) - \lambda_4^{\mathrm{T}} G \in \mathbb{S}_M[z] \tag{19d}$$

where $V^{(k)}$ is given. Clearly, SOSP3 can also be efficiently solved by linear semidefinite programming. The feasible solution $V^{(k+1)}$ is also a feasible solution of SOSP1, which gives a larger estimate of the DA described as $D^{(k+1)} = \mathcal{L}_1(V^{(k+1)}(z))$.

Based on the above analysis, we propose a six-step algorithm, called expanding annular domain algorithm, to compute an estimated DA of power systems as shown in Algorithm 1.

The EAD algorithm can estimate the entire DA rather than the local relevant boundary of the DA for lossy power systems(considering transfer conductance). Among traditional methods, the Closest UEP method combined with numerical energy functions is the only one that can estimate the entire DA for lossy power systems [5], [10]. Hence, we will compare the effectiveness of these two methods in Section V.

Remark 3: In Algorithm 1, a higher $\deg(V)$ will bring a less conservative estimate of the DA, but will increase computational complexity of the corresponding SOS problems [16]. Empirically, to implement the algorithm in Algorithm 1, the degree of the polynomials must satisfy

$$\begin{aligned} \text{SOSP0}: &\deg(V_0) \geq \deg(\lambda_1 G, q_1) \\ &\deg(\dot{V}_0, s_1 p) \geq \deg(\lambda_2 G, q_2) \\ \text{SOSP0'}: &\deg(s_2 \dot{V}_0) \geq \deg(s_1 V_0, \lambda G, q_1) \\ \text{SOSP2}: &\deg(s_4 \dot{V}^{(k)}) \geq \deg(s_2 V^{(k)}, s_3 V^{(k)}, \lambda_3^{\mathrm{T}} G) \\ \text{SOSP3}: &\deg(s_1 V^{(k)}) \geq \deg(V^{(k+1)}, \lambda_1^{\mathrm{T}} G) \\ &\deg(V^{(k+1)}) \geq \deg(\lambda_2^{\mathrm{T}} G, q) \\ &\deg(\bar{s}_4 \dot{V}^{(k+1)}) \geq \deg(\bar{s}_2 V^{(k+1)}, s_3 V^{(k)}, \lambda_3^{\mathrm{T}} G) \\ &\deg(s_5 V^{(k)}) \geq \deg(V^{(k+1)}, \lambda_4^{\mathrm{T}} G) \end{aligned}$$

Remark 4: Our algorithm is totally different from the existing expanding interior algorithm [18] whose idea is

Algorithm 1: Expanding Annular Domain (EAD) algorithm

Input: The degree of all assumed Lyapunov function $\deg(V)$, a positive definite polynomial $p_0(z)$, a positive number γ_0, small positive parameters $\varepsilon_{1,2}$, an empirical parameter $\beta \in (0, 1]$.

Step 0 (a) Set $p = p_0(z)$, $\gamma = \gamma_0$ and solve problem SOSP0. If the problem is feasible, then save the result V_0 as v_0 and go to (b). Otherwise, reset $\deg(V)$, $p_0(z)$, γ_0 and try (a) again.

(b) Set $V_0 = v_0$ and perform a bisection search on c to solve problem SOSP0'. Save the resulting c as c_0. Set $k = 1$, $V^{(1)} = V_0/c_0$ and regard the set $D^{(1)} := \mathcal{L}_1(V^{(1)}(z))$ as an initial estimated DA. Then go to Step 1.

Step 1 Update $V^{(k)}$ and solve problem SOSP2. If the problem is feasible, save the resulting s_2, s_4 as $\bar{s}_2, \bar{s}_4$, respectively, and go to Step 2.

Step 2 Update $V^{(k)}$, $\bar{s}_2$ and $\bar{s}_4$, then solve problem SOSP3. If the problem is feasible, save the resulting $V^{(k+1)}$ and go to Step 3. Otherwise, go to Step 4.

Step 3 Set $k = k + 1$, and we obtain a larger estimated DA $D^{(k)} := \mathcal{L}_1(V^{(k)}(z))(k > 1)$. Then go to Step 1.

Step 4 If $k = 1$, reset $\deg(V)$, $p_0(z)$, γ_0, $\varepsilon_{1,2}$ and β, then go to Step 0. If $k > 1$, save $V^{(k)}(z)$ as $V_d(z)$ and go to Step 5.

Step 5 Output $V_d(h(x))$ by the coordinate transformation (6). And regard the set $D_d := \{x|V_d(h(x)) \leq 1\}$ as the desired largest estimate of the DA.

explained as follows. Let two level sets satisfy $\mathcal{L}_\gamma(p(z)) \subseteq \mathcal{L}_c(V(z))$, then $\mathcal{L}_c(V(z))$, an estimate of the DA, can be expanded by expanding $\mathcal{L}_\gamma(p(z))$. Even though the algorithm has been improved [19], it is still too complex and contains two iteration loops and two SOS optimization problems (not including initialization). In comparison, our algorithm shown in Algorithm 1 contains one iteration loop and two SOS feasibility problems which are easier to solve. Additionally, the expanding interior algorithm is based on conventional Lyapunov stability theory, which might lead to more conservative results according to Remark 2.

IV. Analysis of the Proposed Algorithm

For the proposed algorithm, we discuss the following details.

1) Constraint (17b) also implies that the $D^{(k+1)} = \mathcal{L}_1(V^{(k+1)}(z))$ is bounded and, more precisely, compact. This result, which is necessary according to proposition 1, is proven as follows. From (17b), we have $V^{(k+1)}(z) \geq q(z)$, $\forall z \in \{z|G(z) = 0\}$. Since $q(z)$ is a given positive definite polynomial that is radially unbounded, there exist two K_∞ functions α_1 and α_2 such that $\alpha_1(\|z\|) \leq q(z) \leq \alpha_2(\|z\|)$. Therefore, the inclusion relation $\{z|V^{(k+1)}(z) \leq 1\} \subseteq \{z|q(z) \leq 1\} \subseteq \{z|\alpha_1(\|z\|) \leq 1\}$ holds for all $z \in \{z|G(z) = 0\}$. Since $\{z|\alpha_1(\|z\|) \leq 1\}$ is a bounded set, we can conclude that set $D^{(k+1)} = \mathcal{L}_1(V^{(k+1)}(z)) = \{z|V^{(k+1)}(z) \leq 1, G(z) = 0\}$ is bounded. Moreover, set $D^{(k+1)}$ is also closed because $V^{(k+1)}(z)$ and $G(z)$ are continuously differentiable functions. These two facts prove that set $D^{(k+1)}$ is compact.

2) For the proposed algorithm, if the initialization is successful, there exist solutions to the SOSP2 (Step 1) at every $k \in \{1, 2, \ldots\}$. The proof is given as follows. If the initialization is successful, we have an initial estimated DA $D^{(1)} = \mathcal{L}_1(V^{(1)}(z))$. Then $\dot{V}^{(1)}(z) < 0$ holds for all $z \in \{z|V^{(1)}(z) \leq 1\} \supseteq \{z|\beta \leq V^{(1)}(z) \leq 1\}$, which means the SOSP2 must be solvable at $k = 1$. When $k > 1$, the feasible solution to the SOSP3 at iteration $k-1$ yields that $\dot{V}^{(k)}(z) < 0$ holds for all $z \in \{z|V^{(k)}(z) \leq 1, V^{(k-1)}(z) \geq \beta\}$. From Assumption 1, we have $\{z|V^{(k-1)}(z) \geq \beta\} \supset \{z|V^{(k)}(z) \geq \beta\}$. Hence, $\dot{V}^{(k)}(z) < 0$ holds for all $z \in \{z|\beta \leq V^{(k)}(z) \leq 1\}$. That is, there exist solutions to the SOSP2 at each iteration $k \in \{1, 2, \ldots\}$, which completes the proof.

3) Our algorithm converges if the boundary of the DA is not empty. The proof is provided as follows. In the iteration loop, the estimate of the DA does not become worse since the small parameter $\varepsilon_1 > 0$ in the constraint (17a) guarantees that $D^{(k+1)}$ is always larger than $D^{(k)}$ when k is updated. Meanwhile, ε_1 also affects the rate of expanding the estimate of the DA. Therefore, if $\partial D_{\mathrm{e}} \neq \varnothing$ holds, there exists an iteration $k^* \in \mathbb{Z}_+$ such that $\partial D^{(k^*)} \cap \partial D_{\mathrm{e}} \neq \varnothing$, where D_{e} denotes the exact DA. This result implies that $D^{(k)}$ is asymptotically close to D_{e} as $k \to k^*$, which proves the convergence of our algorithm. For power systems, the boundary of D_{e} is always reachable and thus the proposed algorithm is feasible.

4) The input parameters in Algorithm 1 can be set according to the following rules. Firstly, for power system models, $\deg(V) = 2$ or $\deg(V) = 4$ are proper since searching for higher-order polynomial Lyapunov functions requires greater computational burden. In addition, the set $P = \{z|p_0(z) \leq \gamma_0, \gamma_0 > 0\}$ needs to be contained in the DA to guarantee that the initialization will be successful. Under such a condition, there must be an invariant subset of the DA containing the set P such that V_0 can be found. For example, in our cases, we often choose $p_0(z) = \sum_{i=1}^{M} z_i^2$ and $\gamma_0 = 0.1$. As for small parameter $\varepsilon_{1,2}$, they play an important role in the rate of expanding estimated DAs and can be set to $10^{-6} \sim 10^{-4}$. Finally, parameter β is usually chosen between 0.5 and 0.8, because a large β may limit the search space of $V^{(k+1)}(z)$, while a small β may increase the number of iterations of the algorithm.

V. Examples

A. Estimating the DA for a Two-machine Versus Infinite Bus System

For the purpose of illustration, we first consider a two-machine versus infinite bus system introduced in [8]. Its internal node model can be described by equations (5), where the infinite bus is regarded as the 3rd machine and the last equation in (5) is removed. Parameters of the internal node model are shown as follows:

$$Y = G + \mathrm{i}B$$

$$= \begin{bmatrix} 0.61-4.12\mathrm{i} & 0.09+0.79\mathrm{i} & 0.26+2.94\mathrm{i} \\ 0.09+0.79\mathrm{i} & 0.34-8.45\mathrm{i} & 0.24+7.43\mathrm{i} \\ 0.26+2.94\mathrm{i} & 0.24+7.43\mathrm{i} & 12.64-13.15\mathrm{i} \end{bmatrix}$$
$$M = [0.053, 0.079],\ D = [0.2, 0.2],\ P_m = [2.49, 4.21]$$
$$E = [1.074, 1.057, 1]$$

In this case, the state vector is defined as $[\delta_1, \delta_2, \omega_1, \omega_2]$ and the ASEP is $[0.466, 0.462, 0, 0]$. We shift the ASEP to the origin and obtain a new state vector $x = [\Delta\delta_1, \Delta\delta_2, \omega_1, \omega_2] = [\delta_1 - 0.466, \delta_2 - 0.462, \omega_1, \omega_2]$. Then by coordinate transformation (6), the system is described as (7) and the transformed state vector is $z = [z_1, z_2, z_3, z_4, z_5, z_6]$. Setting $\deg(V) = 2$, $p_0(z) = \sum_{i=1}^{6} z_i^2$, $\gamma_0 = 0.1$, $\varepsilon_1 = 10^{-4}$, $\varepsilon_2 = 10^{-6}$ and $\beta = 0.7$, we obtain an estimate of the DA by the EAD algorithm (Algorithm 1), taking 50 s, and depict its boundary by yellow curve in Fig. 2. Resetting $\deg(V) = 4$, $p_0(z) = \sum_{i=1}^{6} z_i^4$, we obtain another approximated DA boundary by the EAD algorithm (red curve, taking 152 s). Moreover, we compare our results with those provided by the expanding interior algorithm with $\deg(V) = 2$ (blue curve, taking 255 s) [18] and the Closest UEP method with numerical energy functions (the left branch of green curve) [10]. More details are shown in Appendix A.

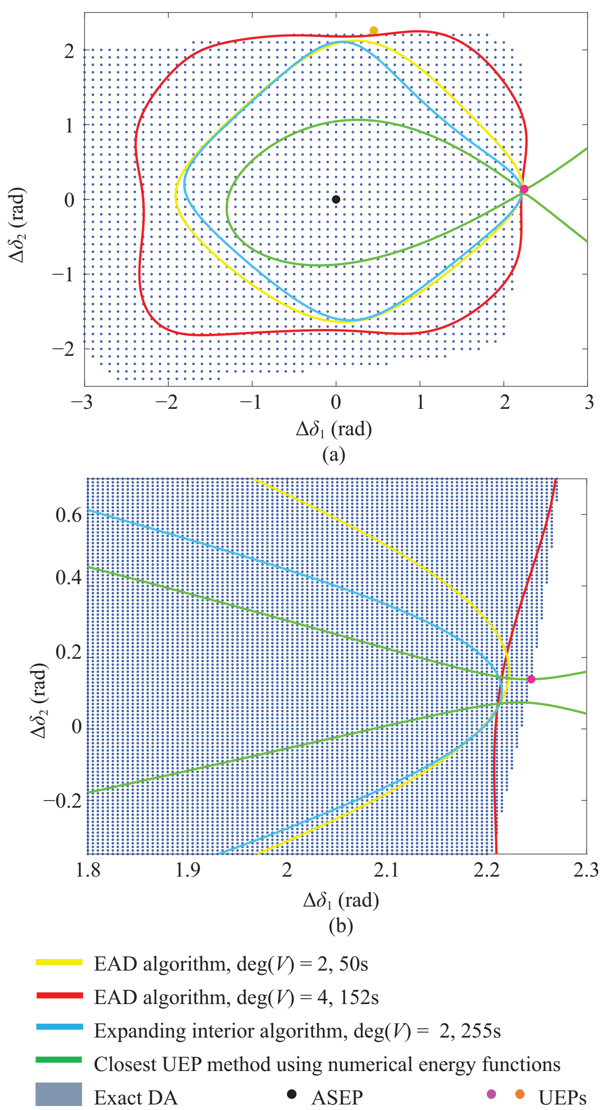

Fig. 2. The estimates of the DA for a two-machine power system projected in the angle space ($\omega_1 = \omega_2 = 0$). (a) Overview. (b) Enlarged area from (a).

From Fig. 2(a), we notice the estimate by the EAD algorithm significantly improves that obtained by the expanding interior algorithm and the traditional Closest UEP method. Additionally, the EAD algorithm gives a better estimate but takes more computation time with $\deg(V) = 4$ than with $\deg(V) = 2$, which confirms the statement in Remark 3. However, the expanding interior algorithm produces a more conservative result and consumes more computation time, which, according to Remark 4, may be caused by excessive iterative steps and excessively strict stability theory. Furthermore, we notice the Closest UEP method gives the most conservative estimated boundary of the DA. It is shown in Fig. 2(b) the result of the Closest UEP method is inaccurate since the Closest UEP (purple point) is not the only intersection of the approximated stability boundary (green curve) and the exact DA boundary, which is not allowed theoretically [5], probably due to the inevitable numerical error of the numerical energy function used. In summary, the proposed EAD algorithm has advantages over existing methods in reducing conservativeness of results and improving calculation speed.

B. Estimating the DA for the Modified IEEE 4-machine-11-bus Power System

We further consider the IEEE 4-machine-11-bus power system [1], where the serial number of generators and the capacity of loads are modified. The parameters of its internal node model (5) are shown as follows:

$$Y = G + \mathrm{i}B$$
$$= \begin{bmatrix} 2.97-10.81\mathrm{i} & 0.78+0.94\mathrm{i} & 1.19+1.40\mathrm{i} & 1.10+6.17\mathrm{i} \\ 0.78+0.94\mathrm{i} & 1.20-9.25\mathrm{i} & 0.42+6.11\mathrm{i} & 0.51+0.63\mathrm{i} \\ 1.19+1.40\mathrm{i} & 0.42+6.12\mathrm{i} & 1.95-10.87\mathrm{i} & 0.78+0.94\mathrm{i} \\ 1.10+6.17\mathrm{i} & 0.51+0.63\mathrm{i} & 0.78+0.94\mathrm{i} & 1.65-9.21\mathrm{i} \end{bmatrix}$$
$$M = [0.310, 0.295, 0.295, 0.310]$$
$$D = [0.18, 0.171, 0.171, 0.18]$$
$$P_m = [5.062, 5.259, 5.293, 5.115]$$
$$E = [1.088, 1.104, 1.167, 1.072]$$

In this case, we estimate the DAs by the EAD algorithm and the Closet UEP method utilizing numerical energy functions [4]. To facilitate finding the UEP and constructing the energy function, we have to consider the relative rotor angular velocity $\omega_{in} = \omega_i - \omega_n$ and simply rewrite equations (5), where $i = 1, 2, \ldots, n-1$. Hence, the state vector is defined as $[\delta_{14}, \delta_{24}, \delta_{34}, \omega_{14}, \omega_{24}, \omega_{34}]$ and the ASEP is $[-0.1272, 0.3417, 0.2022, 0, 0, 0]$. By shifting the ASEP to the origin, we obtain a new state vector $x = [\Delta\delta_{14}, \Delta\delta_{24}, \Delta\delta_{34}, \omega_{14}, \omega_{24}, \omega_{34}] = [\delta_{14}+0.1272, \delta_{24}-0.3417, \delta_{34}-0.2022, \omega_{14}, \omega_{24}, \omega_{34}]$. Then by coordinate transformation (6), the system is described as (7) and the transformed state vector is $z = [z_1, z_2, \ldots, z_9]$. Setting $\deg(V) = 2$, $p_0(z) = \sum_{i=1}^{9} z_i^2$, $\gamma_0 = 1.5$, $\varepsilon_1 = 10^{-4}$, $\varepsilon_2 = 10^{-6}$ and $\beta = 0.7$, we obtain an estimate of the DA by the EAD algorithm and its boundary is shown in Fig. 3. More details are shown in Appendix B.

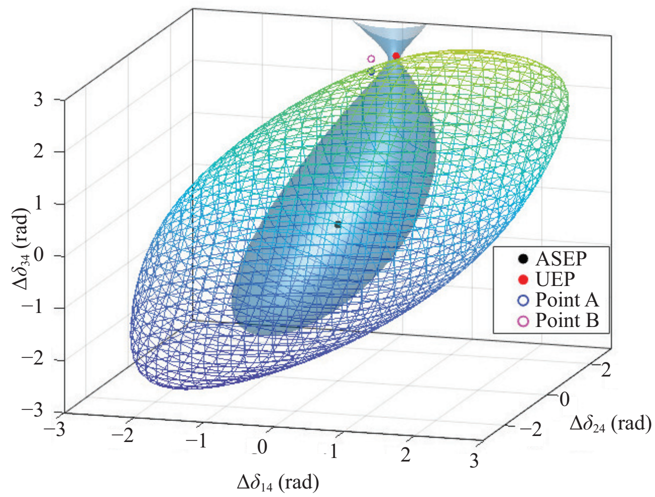

Fig. 3. The estimates of the DA for the IEEE 4-machine-11-bus power system projected in the angle space ($\omega_{14} = \omega_{24} = \omega_{34} = 0$). The outer surface shows the boundary of estimated DA obtained by the EAD algorithm, while the inner surface shows that obtained by the Closest UEP method.

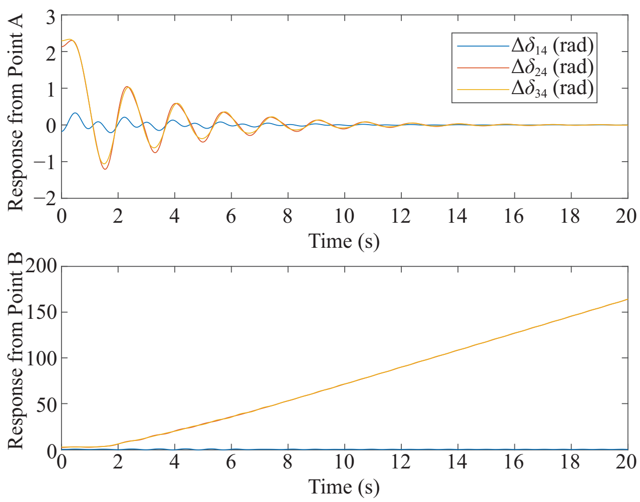

Fig. 4. Time response of the IEEE 4-machine-11-bus system from initial points A and B indicated in Fig. 3.

In Fig. 3, we notice the EAD algorithm gives a very satisfactory approximation of the DA (outer surface), while the Closest UEP method provides an inaccurate and conservative result (inner surface), similar to Example A. Since it is hard to determine the exact DA in this case, we choose two points A $(-0.1765, 2.132, 2.294)$ and B $(-0.1765, 2.132, 2.534)$ to test accuracy of our estimate. Points A and B are very close, located inside and outside the approximated DA given by the EAD algorithm, respectively. It can be seen from Fig. 4 that the time response of the system from initial point A converges to the equilibrium point $x = 0$, while that from initial point B dose not. These results imply that the estimated DA obtained by our method can produce a fairly accurate stability assessment.

VI. Conclusion

This paper has proposed an EAD algorithm to estimate and enlarge the domain of attraction of power systems considering transfer conductance. The algorithm is based on the modified Lyapunov stability theory, where we first introduce the concept of annular domain of attraction and its related stability criterion. With sum-of-squares programming, we enlarge an initial DA by iteratively searching for polynomial Lyapunov functions and determining a series of annular domains of attraction, which leads to a bilinear SOS feasibility problem. For addressing such a problem, the EAD algorithm uses a coordinate-wise iterative method to produce linear SOS programs, which is advantageous over existing algorithms in improving computational speed and reducing conservativeness of results. Furthermore, we have analyzed the algorithm in many details and provided theoretical guarantees for its validity and convergence. Our method has been tested on many multi-machine power systems and to better illustrate our work, we introduce its implementation on two classical power system cases with comparisons to existing methods.

In the future, we would like to refine our work in two directions. First, the EAD algorithm will be extended to estimation of robust DAs of power systems with uncertain parameters and bounded disturbances, addressing the computationally complex problems in existing related studies [26]. Second, due to the worse-case polynomial time complexity of SDP techniques for solving SOS problems, it is quite challenging to directly apply the EAD algorithm to large-scale power systems. Therefore, we will combine our method with the vector Lyapunov function theory [27], [28] or the dissipative system theory [29], [30] to analyze the connective stability [31] of power systems. These attempts are necessary and can overcome the computational difficulties that cannot be settled effectively only by accelerating solutions of SOS problems numerically [32], [33].

Appendix

A. The DAs Estimated by Different Methods in Example A

EAD algorithm: $\deg(V) = 2$, $D(x) = \{x | V(x) \leq 1\}$, where

$$\begin{aligned} V(x) = {} & 0.5324\sin(x_1)^2 - 0.3179(\cos(x_1)-1)(\cos(x_2)-1) \\ & - 0.0169\sin(x_1)\sin(x_2) + 0.5288\sin(x_2)^2 \\ & + 0.0053x_3(\cos(x_1)-1) + 0.0026x_3(\cos(x_2)-1) \\ & + 0.0033x_4(\cos(x_1)-1) + 0.0005x_4(\cos(x_2)-1) \\ & + 0.0010x_3x_4 + 0.0308\sin(x_1)(\cos(x_1)-1) \\ & + 0.0736\sin(x_1)(\cos(x_2)-1) \\ & + 0.0381\sin(x_2)(\cos(x_1)-1) \\ & + 0.0973\sin(x_2)(\cos(x_2)-1) \\ & + 0.0256x_3\sin(x_1) + 0.0020x_3\sin(x_2) \\ & + 0.0030x_4\sin(x_1) + 0.0164x_4\sin(x_2) \\ & + 0.2746(\cos(x_1)-1)^2 + 0.3261(\cos(x_3)-1)^2 \\ & + 0.0047x_3^2 + 0.0042x_4^2 \end{aligned}$$

Expanding interior algorithm [18]: $\deg(V) = 2$, $D(x) = \{x | V(x) \leq c\}$, where

$$V(x) = 0.3481\sin(x_1)\sin(x_2) - 1.227\cos(x_2)$$

$$
\begin{aligned}
&- 0.8683\cos(x_1) + 0.0013(\cos(x_1) - 1)(\cos(x_2) - 1)\\
&+ 1.485\sin(x_1)^2 + 1.35\sin(x_2)^2 + 0.0144x_3(\cos(x_1) - 1)\\
&+ 0.0090x_3(\cos(x_2) - 1) + 0.0367x_4(\cos(x_1) - 1)\\
&- 0.0003x_4(\cos(x_2) - 1) + 0.0077x_3x_4\\
&+ 0.2334\sin(x_1)(\cos(x_1) - 1) + 0.3449\sin(x_1)(\cos(x_2)\\
&- 1) + 0.4098\sin(x_2)(\cos(x_1) - 1)\\
&+ 0.4106\sin(x_2)(\cos(x_2) - 1) + 0.1044x_3\sin(x_1)\\
&- 0.0190x_3\sin(x_2) + 0.0607x_4\sin(x_1) + 0.0664x_4\sin(x_2)\\
&+ 0.6299(\cos(x_1) - 1)^2 + 0.5794(\cos(x_2) - 1)^2\\
&+ 0.0195x_3^2 + 0.0165x_4^2 + 2.0953\\
&c = 3.6344
\end{aligned}
$$

Closest UEP method: the numerical energy function $V(x)$ [4] is obtained by the first-integral principle and ray approximation scheme, an estimated DA is the branch of $D(x) = \{x|V(x) \le c\}$ that contains the ASEP, where

$$
\begin{aligned}
&V(x) = 0.0265x_3^2 + 0.0395x_4^2 + 0.2559\sin(x_2 + 0.462)\\
&- 3.829x_2 - 0.8990\cos(x_1 - x_2 + 0.0044)\\
&- 7.8553\cos(x_2 + 0.462) - 3.1599\cos(x_1 + 0.4664)\\
&- 1.7805x_1 + 0.2838\sin(x_1 + 0.4664)\\
&+ (0.1024\sin(x_1 - x_2 + 0.0044) - 0.0004)\\
&\quad (x_1 + x_2)/(x_1 - x_2) + 10.5116\\
&c = 3.1992
\end{aligned}
$$

Note that the $V(x)$ computed by the EAD algorithm with $\deg(V) = 4$ is not convenient to show here due to its long length.

B. The DAs estimated by different methods in Example B

EAD algorithm: $\deg(V) = 2$, $D(x) = \{x|V(x) \le 1\}$, where

$$
\begin{aligned}
&V(x) = 0.2829\sin(x_1)^2 - 0.1814\sin(x_1)\sin(x_3)\\
&- 0.3356\sin(x_2)\sin(x_3) - 0.1373(\cos(x_1) - 1)\\
&(\cos(x_2) - 1) - 0.1837(\cos(x_1) - 1)(\cos(x_3) - 1)\\
&- 0.4085(\cos(x_2) - 1)(\cos(x_3) - 1)\\
&- 0.1425\sin(x_1)\sin(x_2) + 0.3219\sin(x_2)^2\\
&+ 0.3433\sin(x_3)^2 - 0.0008x_4(\cos(x_1) - 1)\\
&+ 0.0041x_4(\cos(x_2) - 1) + 0.0024x_5(\cos(x_1) - 1)\\
&+ 0.0040x_4(\cos(x_3) - 1) - 0.0075x_5(\cos(x_2) - 1)\\
&+ 0.0050x_6(\cos(x_1) - 1) + 0.0065x_5(\cos(x_3) - 1)\\
&- 0.0004x_6(\cos(x_2) - 1) - 0.0076x_6(\cos(x_3) - 1)\\
&- 0.0094x_4x_5 - 0.0085x_4x_6 + 0.0002x_5x_6\\
&- 0.0219\sin(x_1)(\cos(x_1) - 1)\\
&+ 0.0118\sin(x_1)(\cos(x_2) - 1) + 0.001\sin(x_2)(\cos(x_1) - 1)\\
&- 0.0268\sin(x_1)(\cos(x_3) - 1) + 0.0309\sin(x_2)(\cos(x_2) - 1)\\
&+ 0.0174\sin(x_3)(\cos(x_1) - 1) + 0.0321\sin(x_2)(\cos(x_3) - 1)\\
&- 0.0246\sin(x_3)(\cos(x_2) - 1) + 0.0136\sin(x_3)(\cos(x_3) - 1)\\
&+ 0.0211x_4\sin(x_1) - 0.0151x_4\sin(x_2) + 0.0039x_5\sin(x_1)\\
&- 0.0119x_4\sin(x_3) + 0.0391x_5\sin(x_2) + 0.0002x_6\sin(x_1)\\
&+ 0.0006x_5\sin(x_3) - 0.0022x_6\sin(x_2) + 0.034x_6\sin(x_3)\\
&+ 0.3046(\cos(x_1) - 1)^2 + 0.3479(\cos(x_2) - 1)^2\\
&+ 0.3785(\cos(x_3) - 1)^2 + 0.0077x_4^2 + 0.0085x_5^2 + 0.0076x_6^2.
\end{aligned}
$$

Closest UEP method: the numerical energy function $V(x)$ is given in [4] (See section 2.3.1) and is too complicated to show here. Note that we use the ray approximation scheme to compute the path-dependent terms and utilize the relative rotor angular velocity formulation to guarantee that UEPs exist when $\omega_{14} = \omega_{24} = \omega_{34} = 0$. An estimated DA is $D(x) = \{x|V(x) \le c\}$, where $c = 4.9899$.

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